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EDGE ORIENTATIONS ON CUBIC GRAPHS WITH A MAXIMUM NUMBER OF PAIRS OF OPPOSITELY ORIENTED EDGES

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We consider the problem: Characterize the edge orientations of a finite graph with a maximum number of pairs of oppositely oriented edges. The problem is solved for finite cubic graphs.

An undirected graph X consists of a non-empty set $V(X)$, the vertices of X, and a set $E(X)$ of unordered pairs of distinct elements of $V(X)$, the edges of X. An *edge orientation* of X is a mapping $\mathcal{O}: E(X) \to V^2(X)$, which assigns to each edge one of its endvertices as its first vertex and the other as its second vertex. Of course each edge orientation of X yields a directed graph and vice versa. We call two edges of X *oppositely oriented*, if both have the same first vertex or the same second vertex. By $e(X, \mathcal{O})$ we denote the number of pairs of oppositely oriented edges in X according to the edge orientation $\mathcal{O}$ of X. We call an edge orientation $\mathcal{O}$ *e-maximal*, if $e(X, \mathcal{O})$ has maximum value among all edge orientations of X. We consider the problem: characterize the e-maximal edge orientations of a graph X. We give such a characterization for finite cubic graphs.

It suffices to consider connected graphs. Therefore in the following we denote by X a finite, connected, cubic graph with $2n$ vertices. Let $d^-(u)$ be the indegree, $d^+(u)$ the outdegree of a vertex u in a directed graph and $\delta(G)$ the minimal degree of an undirected graph G. For further terminology see [1] or [2].

If X is provided with an edge orientation $\mathcal{O}$, we call a vertex $u \in V(X)$ an

emitter (E)	if	$d^+(u)=3, d^-(u)=0$
receiver (R),	if	$d^+(u)=0, d^-(u)=3,$
confluence vertex (C),	if	$d^+(u)=1, d^-(u)=2$
branching vertex (B),	if	$d^+(u)=2, d^-(u)=1.$

An ER-edge is an edge, the first vertex of which is an emitter and the second vertex of which is a receiver. Analogously EC-, EB-, ... edges are defined.

As each vertex of X yields at least one pair of oppositely oriented edges, but at most three such pairs, we obtain for each edge orientation $\mathcal{O}$ of X the following inequality:

$$2n \leq e(X, \mathcal{O}) \leq 6n$$

It is easy to show, that there exists to each X an edge orientation $\mathcal{O}$ with

$e(X, \mathcal{O}) = 2n$; whereas $e(X, \mathcal{O}) = 6n$ is only possible if each vertex of X yields three pairs of oppositely oriented edges, i.e. if each vertex of X is an emitter or a receiver. An edge orientation of that kind obviously exist if and only if X is a bipartite graph. Otherwise always $e(X, \mathcal{O}) < 6n$. The e-maximal edge orientations of X will be characterized in the following.

We call a subgraph F of X a *framework*, if

(1) F is spanning.

(2) F is maximal bipartite (i.e. the adjunction of any edge of $E(X) - E(F)$ produces a non-bipartite graph).

(3) $\delta(F) \geq 2$.

Clearly such a subgraph F is connected.

Theorem 1. *Each finite, connected, cubic graph X contains a framework.*

Proof. The theorem could be deduced immediately from the following theorem, mentioned in [1]: Each graph G contains a bipartite spanning subgraph H such that $d_H(v) \geq \frac{1}{2} d_G(v)$ for all $v \in V$. But we give another proof, because the details are important for the following.

We provide X with an e-maximal edge orientation. We show, that there does not exist a CC-edge in X. For if there exists a CC-edge in X, we reverse the orientation of this edge. Thereby $\mathcal{O}$ turns to an edge orientation $\mathcal{O}'$ of X. It is easy to calculate, that $e(X, \mathcal{O}') = e(X, \mathcal{O}) + 2 > e(X, \mathcal{O})$, contrary to the assumption that $\mathcal{O}$ is e-maximal. Analogously it can be shown, that there exists neither a CB- nor a BB-edge in X. Hence in X only ER-, EC-, BR-, BC-, CR- or EB-edges can appear.

Furthermore, we show, that there are no adjacent CR-edges in X. Two adjacent CR-edges are bound to have the same receiver as second vertex. Reversing the orientations of both the adjacent CR-edges, $\mathcal{O}$ turns to an edge orientation $\mathcal{O}''$ of X. It is easy to calculate, that $e(X, \mathcal{O}'') = e(X, \mathcal{O}) + 2 > e(X, \mathcal{O})$, contrary to the assumption, that $\mathcal{O}$ is e-maximal. Analogously it can be shown, that there are no adjacent EB-edges in X.

Let F be the subgraph of X induced by all ER-, EC-, BR-, and BC-edges. As a vertex of X is incident with at most one edge of the set of the CR-edges and EB-edges, each vertex of X is incident with at least two edges of F. Hence F is a spanning subgraph of X and $\delta(F) \geq 2$. F is bipartite; for if V_1 is the set of all emitters and branching vertices of X and V_2 the set of all receivers and confluence vertices of X, then (V_1, V_2) is a bipartition of F. Moreover, F is maximal bipartite, because each edge of $E(X) - E(F)$ is an EB- or CR-edge, hence joining two vertices of V_1 or two vertices of V_2. □

We say, a framework F of X is *properly oriented* by an edge orientation $\mathcal{O}$ of X, if any two adjacent edges of F are oppositely oriented. Thus we get a bipartition (V_1, V_2) of F with $V_1 = \{u \in V(F) \mid d_F^-(u) = 0\}$ and $V_2 = \{u \in V(F) \mid d_F^+(u) = 0\}$.

Theorem 2. *To each edge orientation $\mathcal{O}$ of X there is at most one framework of X, which is properly oriented by $\mathcal{O}$.*

Proof. Let us assume, there are two different frameworks F_1, F_2 of X, which are both properly oriented by $\mathcal{O}$. Let x be an edge with the first vertex u and the second vertex v and $x \in E(F_1)$, $x \notin E(F_2)$. Let a, b be the neighbouring edges of x incident with u and c, d the neighbouring edges of x incident with v. As $x \in E(F_1)$ and $\delta(F_1) \geq 2$, it follows, that $a \in E(F_1)$ or $b \in E(F_1)$. Without loss of generality let us assume $a \in E(F_1)$. As $x \notin E(F_2)$ and $\delta(F_2) \geq 2$ as well, it follows, that $a \in E(F_2)$ and $b \in E(F_2)$. From x, $a \in E(F_1)$ and a, $b \in E(F_2)$ it follows, that u is first vertex of the edges a and b. Analogously we can show, that v is second vertex of the edges c and d.

If (V_1, V_2) is the bipartition mentioned above, then $u \in V_1$ and $v \in V_2$. Consequently the graph $F_2 + x$ is bipartite, contrary to the assumption that F_2 is maximal bipartite. □

We call an edge orientation $\mathcal{O}$ an *f-orientation* of X, if there exists a framework F of X, which is properly oriented by $\mathcal{O}$. From Theorem 2 it follows, that there exists to each f-orientation $\mathcal{O}$ exactly one framework F of X, which is properly oriented by $\mathcal{O}$. Let us call it the framework belonging to the orientation $\mathcal{O}$.

Lemma 1. *Each e-maximal edge orientation of X is an f-orientation of X.*

Proof. Let $\mathcal{O}$ be an e-maximal edge orientation of X and F the framework constructed in the proof of Theorem 1. Any two adjacent edges of F are oppositely oriented. For according to the definition of F each emitter or branching vertex of X is only first vertex of edges in F and each receiver or confluence vertex of X is only second vertex of edges in F. Hence F is properly oriented by $\mathcal{O}$ and consequently $\mathcal{O}$ is an f-orientation of X. □

Lemma 2. *Let $\mathcal{O}$ be an f-orientation of X and F the framework belonging to $\mathcal{O}$. Let $q(F)$ be the number of edges of F. Then*

$$e(X, \mathcal{O}) = 2 \cdot q(F).$$

Proof. Let $\mathcal{O}$ be an f-orientation of X and F the framework belonging to $\mathcal{O}$. Let F contain f_2 vertices of degree 2 and f_3 vertices of degree 3. By f_2^{E}, f_3^{E}, respectively, we denote the number of vertices of degree 2, 3, respectively, of F, which are emitters in X. Analogously f_2^{R}, f_3^{R}, f_2^{C}, f_3^{C}, f_2^{B}, f_3^{B} are defined. As each vertex of degree 3 of F is an emitter or a receiver in X, it follows, that $f_3^{\mathrm{C}} = f_3^{\mathrm{B}} = 0$.

Let (V_1, V_2) be the bipartition of F mentioned above. As F is maximal bipartite, each edge of $E(X) - E(F)$ must necessarily join two vertices of V_1 or two vertices of V_2 (this is also true, if $X = F$). I.e. each edge of $E(X) - E(F)$ is an

EB- or CR-edge. As each vertex of degree 2 of F is incident either with exactly one of these EB-edges or with exactly one of these CR-edges, the following equations are valid:

$$f_2^{\mathrm{E}} = f_2^{\mathrm{B}}, \tag{1}$$

$$f_2^{\mathrm{R}} = f_2^{\mathrm{C}}, \tag{2}$$

$$2(f_2^{\mathrm{E}} + f_2^{\mathrm{R}}) = f_2. \tag{3}$$

Applying (1), (2), (3) we calculate

$$\begin{aligned} e(X, \mathcal{O}) &= 3f_3^{\mathrm{E}} + 3f_3^{\mathrm{R}} + 3f_2^{\mathrm{E}} + 3f_2^{\mathrm{R}} + f_2^{\mathrm{C}} + f_2^{\mathrm{B}} \\ &= 3(f_3^{\mathrm{E}} + f_3^{\mathrm{R}}) + 4(f_2^{\mathrm{E}} + f_2^{\mathrm{R}}) \\ &= 3f_3 + 2f_2 = 2 \cdot q(F). \quad \square \end{aligned}$$

As one can show by examples, two frameworks of X must not have the same number of edges. We call a framework F of X a *greatest framework of* X, if it has a maximum number of edges among all frameworks of X.

Theorem 3. *An edge orientation $\mathcal{O}$ of X is e-maximal if and only if $\mathcal{O}$ is an f-orientation of X and the framework belonging to $\mathcal{O}$ is a greatest framework of X.*

Proof. According to Lemma 1 each e-maximal edge orientation of X is an f-orientation of X. Each f-orientation with a greatest framework of X trivially is an f-orientation of X. Therefore we only need to investigate f-orientations of X.

To each f-orientation $\mathcal{O}$ of X there exists a framework F, the framework belonging to $\mathcal{O}$, such that

$$e(X, \mathcal{O}) = 2 \cdot q(F) \tag{1}$$

according to Lemma 2. On the other hand to each framework F of X there obviously exists an f-orientation $\mathcal{O}$ of X, properly orienting F, such that (1) holds. From this it follows, that $e(X, \mathcal{O})$ has maximum value among all f-orientations of X if and only if $q(F)$ has maximum value among all frameworks of X. $\square$

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London, 1976).
[2] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1972).